QFT in curved spacetimes and analogue gravity
Lecture 2
Particle creation by the expansion of the universe: the frequency-mixing mechanism

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Introduction and motivation: historical context

- Schwinger's analysis of pair creation in electric fields dates back to 1951, a period during which quantum electrodynamics (QED) was flourishing.
- In the late 1950s and early 1960s, Feynman and DeWitt turned their attention to the problem of quantizing the gravitational field, hoping to mirror the success of QED.
- Their efforts did not succeed, and despite remarkable progress since then, the full quantization of gravity remains an open problem
- It is worth noting, however, that this line of research did lead to an important triumph: the successful perturbative quantization of Yang-Mills fields.

Introduction and motivation: historical context

- Confronted with the impasse in quantum gravity, Parker (1963-66) introduced the idea of studying quantum fields in a space-time that itself satisfies Einstein's gravitational field equations.

  At the time, I felt that quantizing the nonlinear gravitational field itself was so difficult that I would not be able to make significant progress in trying to go beyond the deep work that had already been done in that area. Nevertheless, I felt that it would be valuable to study quantized elementary particle fields in the curved space-times that were solutions of the non-linear Einstein gravitational field equations.

  I started by looking for new consequences of quantum field theory in the isotropically expanding cosmological space-times that were solutions of the nonlinear equations of general relativity.

  More details in

  A. Ferreiro, J. N-S and S. Pla, "The Birth of Gravitational Particle Creation ... ": [arXiv:2511.13518] To appear in The European Physical Journal H

Introduction and motivation: historical context

- As in the Schwinger effect, where the electric field is treated as a classical background, we may likewise treat the gravitational field as a classical entity. This is the framework we will explore in this lecture.

- The simplest and most tractable scenario is to consider an isotropic and homogeneous space-time, characterized solely by a cosmic scale factor $a(t)$

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

Field equation

- Consider the simplest form of an expanding universe

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 \equiv g_{\mu\nu}dx^\mu dx^\nu$$

and a free scalar field ϕ obeying the minimally coupled KG equation

$$(\Box\phi + m^2)\phi = 0$$

where

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$$

- The field equation becomes

$$a^{-3}\partial_t(a^3\partial_t\phi) - a^{-2}\sum_{i=1}^{3}\partial_i^2\phi + m^2\phi = 0$$

Field equation and canonical quantization

- The action functional is given by

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2)$$

- We want to implement the canonical quantization. [The canonical momentum is $\pi = a^3\dot{\phi}$.]
- Impose equal-time canonical commutation relations:

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\delta^3(\mathbf{x}-\mathbf{x}')$$

$$[\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] = 0 = [\pi(\mathbf{x},t),\pi(\mathbf{x}',t)]$$

- K-G scalar product (the 3-dimensional volume over a constant t hypersurface is $d^3x\sqrt{|g|}$)

$$(\phi_1,\phi_2) = i\int_{t=t_0} d^3x a^3(\phi_1^*\dot{\phi}_2 - \phi_2\dot{\phi}_1^*)$$

- To quantize the field ϕ in HEISENBERG PICTURE $\Rightarrow$ split a generic solution into orthogonal solutions, like “positive” and “negative” frequency solutions in Minkowski space.

$$\phi(t,\vec{x}) = \int d^3k\,[A_{\vec{k}}f_{\vec{k}}(t,\vec{x}) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(t,\vec{x})]\,,$$

Field equation and canonical quantization

- We will take advantage of the 3-spatial homogeneity of the metric.
- We will assume the following ansatz for the modes [$k \equiv |\vec{k}|$, $\sqrt{2(2\pi)^3 a^3(t)}$ is introduced for normalization (see later). For $a = 1$ we recover the normalization used in Minkowski space]

$$f_{\vec{k}}(t, \vec{x}) = \frac{1}{\sqrt{2(2\pi)^3 a^3(t)}} e^{i\vec{k}\vec{x}} h_k(t)$$

- The function $h_k(t)$ satisfies the second-order differential equation [Exercise]

$$\ddot{h}_k + [\omega_k^2 - \frac{3}{4}(\frac{\dot{a}}{a})^2 - \frac{3}{2}\frac{\ddot{a}}{a}]h_k = 0$$

where $\omega_k^2 = k^2/a^2 + m^2$. Note: physical momentum $\vec{p} = \vec{k}/a(t)$. Comoving momentum $\equiv \vec{k}$. Physical length $\vec{y} \equiv \vec{x}a(t)$, so $\vec{y}\vec{p} = \vec{x}\vec{k}$

- Note: the Wronskian condition for the above equation is

$$\frac{d}{dt}[h_k^* \dot{h}_k - \dot{h}_k^* h_k] = 0$$

Normalization

- The functions $h_k(t)$ should obey a normalization condition with respect to the KG inner product

$$(\phi_1, \phi_2) = i \int d^3x a^3 (\phi_1^* \dot{\phi}_2 - \phi_2 \dot{\phi}_1^*)$$

- With the normalization

$$(f_{\vec{k}}, f_{\vec{k}'}) = \delta^3(\vec{k} - \vec{k}')$$
$$(f_{\vec{k}}, f^*_{\vec{k}'}) = 0 \ ,$$

one gets

$$[A_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') \ , \quad [A_{\mathbf{k}}, A_{\mathbf{k}'}] = 0 = [A^\dagger_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}] \ .$$

- This is ensured with the following condition for $h_k(t)$

$$h_k^* \dot{h}_k - \dot{h}_k^* h_k = -2i \ .$$

The fundamental question

- FUNDAMENTAL QUESTION: We have to find a criterion to choose the appropriate form of $h_k(t)$.

- In Minkowski space the natural choice is

$$h_k(t) = \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k t}$$

- In the general case (arbitrary $a(t)$) there is no natural choice for $h_k(t)$.
- We can only assume an asymptotic condition (Einstein's equivalence principle)

$$h_k(t) \sim_{k\to\infty} \frac{1}{\sqrt{\omega_k}} e^{-i\int^t \omega_k(t')dt'} \ ,$$

  where $\omega_k(t) = \sqrt{\frac{k^2}{a^2(t)} + m^2}$
  At short distances, or equivalently, at high energy, the quantization of a field should look like the quantization in Minkowski space
- It is illustrative to reconsider this issue in Minkowski space

The fundamental problem in Minkowski space

- In Minkowski space we can expand a quantized scalar field in terms of many different sets of field modes

$$\phi(t,\vec{x}) = \int d^3p (A_{\vec{p}}\phi_{\vec{p}}(t)e^{i\vec{p}\vec{x}} + A^\dagger_{\vec{p}}\phi^*_{\vec{p}}(t)e^{-i\vec{p}\vec{x}}) = \int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}}(a_{\vec{p}}e^{-ipx} + a^\dagger_{\vec{p}}e^{ipx}) \, ,$$

where (assuming isotropy)

$$\phi_{\vec{p}}(t) = \frac{1}{\sqrt{2(2\pi)^3\omega_p}}(\alpha_p e^{-i\omega_p t} + \beta_p e^{i\omega_p t}) \, .$$

- The normalization of the modes with the KG inner scalar product requires

$$|\alpha_p|^2 - |\beta_p|^2 = 1 \, .$$

- We get the following (Bogoliubov) transformation

$$a_{\vec{p}} = \alpha_p A_{\vec{p}} + \beta^*_p A^\dagger_{-\vec{p}}$$

$$a^\dagger_{\vec{p}} = \alpha^*_p A^\dagger_{\vec{p}} + \beta_p A_{-\vec{p}} \, .$$

$$[A_{\mathbf{p}}, A^\dagger_{\mathbf{p}'}] = \delta^3(\mathbf{p} - \mathbf{p}') \, , \quad [A_{\mathbf{p}}, A_{\mathbf{p}'}] = 0 = [A^\dagger_{\mathbf{p}}, A^\dagger_{\mathbf{p}'}] \, .$$

The fundamental question in Minkowski space

- Define the vacuum state $|0)_{\alpha,\beta}$ by the condition

$$A_{\vec{p}}|0)_{\alpha,\beta} = 0$$

- The usual vacuum $|0\rangle$, is defined by

$$a_{\vec{p}}|0\rangle = 0$$

- Is there any way to distinguish a unique vacuum state ?
- In Minkowski space, the choice of vacuum is natural and unambiguous. We can require:
  1. $\hat{H}|0)_{\alpha,\beta} \propto |0)_{\alpha,\beta}$
  2. $(0|\mathcal{H}|0)_{\alpha,\beta}$ takes a minimun value
  3. The splitting $\phi_{\vec{p}}(t)e^{i\vec{p}\vec{x}}$ versus $\phi^*_{\vec{p}}(t)e^{-i\vec{p}\vec{x}}$ is Lorentz invariant
- Each of the above requirements univocally identifies the state $|0\rangle$ [$\beta_p = 0$] as the vacuum state of the theory

The fundamental problem in Minkowski space

- For instance, consider

$$\begin{aligned}(0|\mathcal{H}|0)_{\alpha,\beta} - \langle 0|\mathcal{H}|0\rangle &= \frac{1}{2}\int d^3p(|\dot{\phi}_{\vec{p}}(t)|^2 + \omega_p^2|\phi_{\vec{p}}(t)|^2 - \frac{\omega_p}{(2\pi)^3})\ . \\ &= \frac{1}{2(2\pi)^3}\int d^3p\ \omega_p(|\alpha_p|^2 + |\beta_p|^2 - 1) \\ &= \frac{1}{(2\pi)^3}\int d^3p\ \omega_p|\beta_p|^2\ .\end{aligned}$$

- This quantity is minimized for $\beta_p = 0$.
- This implies that, up to an irrelevant phase, $\alpha_p = 1$.
- Therefore, the conventional vacuum state $|0\rangle = |0)_{\alpha=1,\beta=0}$ is just *the state of lowest energy*.
- This argument singles out the state $|0\rangle$ as the ground state of the theory, an observation rarely noted in textbooks.

  For fermions, see the appendix in [S. Nadal-Gisbert, J. N-S. and S. Pla, Phys. Rev. D **107**, 085018 (2023)]
- Furthermore, $|0\rangle$ is the only vacuum state possessing Lorentz invariance

Not a single vacuum, but many vacuums

- Any choice of function $h_k(t)$, obeying the asymptotic condition, defines an admissible vacuum state

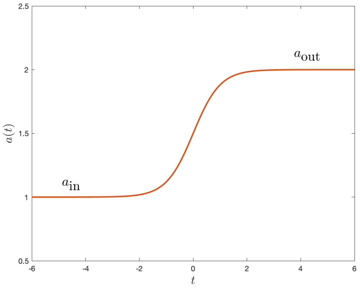
- Not one quantum vacuum, but many quantum vacuums

- Not a single family of inertial frames (related by Lorentz transformations, as in special relativity), but many families of freely falling frames (as in general relativity)

Not a single vacuum: an illustrative example

- For simplicity, assume that the expansion factor $a(t)$ has a time dependence that asymptotically approaches constant values at early and late times of cosmic time

$$
\begin{aligned}
a(t) &\sim a_{in} \quad t \to -\infty \\
a(t) &\sim a_{out} \quad t \to +\infty
\end{aligned}
$$



- Remark: One could just as well replace the intervals in which $a(t)$ is constant with periods of slow expansion

Two natural set of modes

- We have two different set of modes
- Those that, for $t \to -\infty$, behave as

$$h_k^{in}(t) \sim \frac{1}{\sqrt{\omega_{in}}} e^{-i\omega_{in} t} \ ,$$

  $\omega_{in} = \sqrt{(k/a_{in})^2 + m^2}$
  We call them IN modes
- And those that, for $t \sim +\infty$, behave as

$$h_k^{out}(t) \sim \frac{1}{\sqrt{\omega_{out}}} e^{-i\omega_{out} t} \ .$$

  $\omega_{out} = \sqrt{(k/a_{out})^2 + m^2}$.
  We call them OUT modes
- Note: both $h_k^{in}(t)$ and $h_k^{out}(t)$ satisfy the large k asymptotic condition
- In the asymptotic regions, the theory reduces to conventional quantum field theory in Minkowski spacetime.

Notation and Remarks

- IN modes

$$f_{\vec{k}}(\vec{x},t) = \frac{1}{\sqrt{2(2\pi)^3 a^3(t)}} h_k^{in}(t) e^{i\vec{k}\vec{x}} \sim \frac{1}{\sqrt{2(2\pi)^3 a_{in}^3 \omega_{in}}} e^{i(\vec{k}\vec{x}-\omega_{in} t)} \qquad t \to -\infty \, ,$$

- OUT modes

$$g_{\vec{k}}(\vec{x},t) = \frac{1}{\sqrt{2(2\pi)^3 a^3(t)}} h_k^{out}(t) e^{i\vec{k}\vec{x}} \sim \frac{1}{\sqrt{2(2\pi)^3 a_{out}^3 \omega_{out}}} e^{i(\vec{k}\vec{x}-\omega_{out} t)} \quad t \to +\infty \, ,$$

  where $\omega_{in} = \sqrt{(k/a_{in})^2 + m^2}$ and $\omega_{out} = \sqrt{(k/a_{out})^2 + m^2}$.

- Remark: One could equally well replace the periods when $a(t)$ is constant by periods of slow expansion, like $a(t) \sim t^{1/2}, t^{2/3}$ when one can replace

$$\frac{1}{\sqrt{2(2\pi)^3 a_{out}^3 \omega_{out}}} e^{i(\vec{k}\vec{x}-\omega_{out} t)} \quad t \to +\infty$$

  by

$$\frac{1}{\sqrt{2(2\pi)^3 a^3(t) \omega_k(t)}} e^{i(\vec{k}\vec{x}-\int^t dt' \omega_k(t'))} \quad t \to +\infty$$

Two field modes expansions and two vacua

- We have found two natural mode expansions

$$\phi(t,\vec{x}) = \int d^3k [A_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + A^{\dagger}_{\vec{k}} f^{*}_{\vec{k}}(t,\vec{x})]$$

$$\phi(t,\vec{x}) = \int d^3k [a_{\vec{k}} g_{\vec{k}}(t,\vec{x}) + a^{\dagger}_{\vec{k}} g^{*}_{\vec{k}}(t,\vec{x})]$$

- $f_{\vec{k}}(t,\vec{x})$ defines the quantization at early times
- The initial vacuum is defined as

$$A_{\vec{k}}|0_{in}\rangle = 0$$

- $g_{\vec{k}}(t,\vec{x})$ defines the quantization at late times
- The vacuum at late times is defined as

$$a_{\vec{k}}|0_{out}\rangle = 0$$

Parker's mechanism: frequency mixing

- IMPORTANT POINT: the time evolution implies frequency mixing

  Positive-frequency modes $e^{-i\omega_{in} t}$ **evolve into a superposition** *of positive* $e^{-i\omega_{out} t}$ and *negative frequency modes* $e^{+i\omega_{out} t}$

$$h_k^{in}(t) = \alpha_k h_k^{out}(t) + \beta_k h_k^{out*}(t)$$

- Equivalently

$$f_{\vec{k}}(\vec{x}, t) = \alpha_k g_{\vec{k}}(\vec{x}, t) + \beta_k g^*_{-\vec{k}}(\vec{x}, t) \ ,$$

- The coefficients β's and α's are time-independent and obey the identity

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

Parker's mechanism: frequency mixing and pair creation

- IN and OUT vacuum states

$$A_{\vec{k}}|0_{in}\rangle = 0 \qquad a_{\vec{k}}|0_{out}\rangle = 0$$

- The mixing of frequencies is parametrized by a set of coefficients that, in turn, defines what is known as a Bogolubov transformation

$$a_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta_k^* A^\dagger_{-\vec{k}}$$
$$a^\dagger_{\vec{k}} = \alpha_k^* A^\dagger_{\vec{k}} + \beta_k A_{-\vec{k}}$$

Exercise: check it

- The IN vacuum is perceived as a collection of particles at late times (i.e., excitations of the OUT vacuum).
- The mean particle number of the $\vec{k}$ out-mode is given by

$$\langle 0_{in}|N^{out}_{\vec{k}}|0_{in}\rangle = \langle 0_{in}|a^\dagger_{\vec{k}} a_{\vec{k}}|0_{in}\rangle = |\beta_k|^2 \delta^3(\vec{k}-\vec{k})$$

Parker's mechanism: frequency mixing and pair creation

- Number density (per unit physical volume $Va_{out}^3 = (2\pi)^3\delta^3(\vec{0})a_{out}^3$) of created out-particles in mode $\vec{k}$

$$n_{\vec{k}}^{out} \equiv \frac{1}{Va_{out}^3}\langle 0_{in}|N_{\vec{k}}^{out}|0_{in}\rangle = \frac{1}{(2\pi)^3 a_{out}^3}|\beta_k|^2$$

- This shows unambiguously that particles have been spontaneously created out of the vacuum by the changing scale factor of the dynamic universe.
- The total particle number per physical unit volume (summed over all modes $\vec{k}$) is

$$n^{out} = \frac{1}{(2\pi)^3 a_{out}^3}\int d^3k|\beta_k|^2 = \frac{1}{(2\pi)^3}\int d^3p_{out}|\beta_{p_{out}}|^2$$

- Remark: n^{out} is an adiabatic invariant.
- The density of created particles approaches zero if the Hubble rate $\dot{a}/a$ is negligible at each instant of time even if the final amount of expansion a_{out}/a_{in} is large.

Probability distribution of created particles by the expanding universe

- As discussed in previous slices

$$A_{\vec{k}}|0_{in}\rangle = 0 \qquad a_{\vec{k}}|0_{out}\rangle = 0$$

$$a_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta_k^* A^{\dagger}_{-\vec{k}}$$

$$a^{\dagger}_{\vec{k}} = \alpha_k^* A^{\dagger}_{\vec{k}} + \beta_k A_{-\vec{k}}$$

- Furthermore, taking into account that

$$a_{\vec{k}}|0_{in}\rangle = \beta_k^* A^{\dagger}_{-\vec{k}}|0_{in}\rangle = \beta_k^* (\alpha_k^*)^{-1} a^{\dagger}_{-\vec{k}}|0_{in}\rangle$$

we can obtain the probability amplitude of creating one pair of particles, one with momentum $\vec{k}$ and the other with momentum $-\vec{k}$ $[(1(\vec{k}), 1(-\vec{k})]$

$${}_{out}\langle -\vec{k}, \vec{k}|0_{in}\rangle = \langle 0_{out}|a_{-\vec{k}} a_{\vec{k}}|0_{in}\rangle = \frac{\beta_k^*}{\alpha_k^*}\langle 0_{out}|a_{-\vec{k}} a^{\dagger}_{-\vec{k}}|0_{in}\rangle = \frac{\beta_k^*}{\alpha_k^*}\langle 0_{out}|0_{in}\rangle \delta^3(\vec{0})$$

The particles are created in pairs, with total momentum $= \overrightarrow{0}$

Probability distribution of created particles by the expanding universe

- The probability of creating one pair of particles, one with momentum $\vec{k}$ and the other with momentum $-\vec{k}$ $[(1(\vec{k}), 1(-\vec{k})]$ is

$$|_{out}\langle -\vec{k}, \vec{k}|0_{in}\rangle|^2 = \frac{|\beta_k|^2}{|\alpha_k|^2} |\langle 0_{out}|0_{in}\rangle|^2 [\delta^3(\vec{0})]^2$$

  In general, for n pairs

$$|_{out}\langle n(-\vec{k}), n(\vec{k})|0_{in}\rangle|^2 = \frac{|\beta_k|^{2n}}{|\alpha_k|^{2n}} |\langle 0_{out}|0_{in}\rangle|^2 [\delta^3(\vec{0})]^{2n}$$

- Vacuum persistence amplitude

$$\langle 0_{out}|0_{in}\rangle$$

- Another fundamental quantity is the relative probability of observing a pair of particles

$$\frac{|\beta_k|^2}{|\alpha_k|^2}$$

  Note that $\alpha_k \neq 0$

Probability distribution of created particles by the expanding universe

- In general, one can write (the factor $1/2$ is to avoid overcounting)

$$|0_{in}\rangle \quad = \quad \langle 0_{out}|0_{in}\rangle \prod_{\vec{k}} \exp\left(\frac{1}{2}\beta_k^* \alpha_k^{*-1} a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger\right)|0_{out}\rangle \ .$$

- Particles are created in (correlated/entangled) pairs of opposite momenta, and the production of pairs in different modes are uncorrelated events
- Unitarity implies [we also use $|\alpha_k|^2 - |\beta_k|^2 = 1$. We omit the details]

$$|\langle 0_{out}|0_{in}\rangle|^2 = \prod_{\vec{k}} |\alpha_k|^{-1} \leq 1$$

- Remark 1: The *in* and *out* Fock spaces are related by a unitary transformation U. In particular $|0_{in}\rangle = U|0_{out}\rangle$. U can be viewed as a scattering operator $U \equiv S$.

$$U = \prod_{\vec{k}} \exp[\frac{1}{2}(\beta_k^* \alpha_k^{*-1} a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger - \beta_k \alpha_k^{-1} a_{\vec{k}} a_{-\vec{k}})]$$

- Remark 2: U takes the form of a multi-mode squeezed operator, and the $|0_{in}\rangle$ vacuum is also a squeezed vacuum state, in the language of quantum optics.

An illustrative example

- Consider the following prototype model

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2)$$

with

$$a^2(\eta) = 1 + B(1 + \tanh \rho\eta)$$

- Note that, with $dt = a(\eta)d\eta$

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2)$$

- The expansion factor has smoothly shifted from

$$a_{in} \equiv a(-\infty) = 1 \qquad \rightarrow \qquad a_{out} \equiv a(+\infty) = \sqrt{1 + 2B}$$

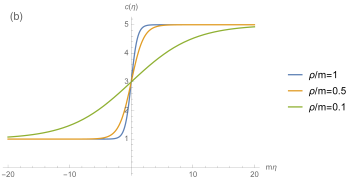


Figura: Time-dependence of the conformal scale factor $C(\eta) \equiv a^2(\eta)$ on ρ/m. The adiabatic limit corresponds to $\rho \rightarrow 0$.

- Consider, for simplicity, $\xi = 1/6$ (this is called the conformal coupling to be added to the lagangian ξR), and $m^2 > 0$.
- By solving the field equations for the modes and imposing the *in* and *out* boundary conditions we can calculate [using special functions and asymptotic properties (we omit details)]

$$\alpha_k = \ldots \qquad \beta_k = \ldots$$

From this we predict the number density of created particles at late times

- The number density of created particles with momentum $\vec{k}$ is given by

$$n_{\vec{k}}^{out} \propto |\beta_k|^2 = \frac{\sinh^2(\pi \frac{\omega_-}{\rho})}{\sinh(\pi \frac{\omega_{in}}{\rho}) \sinh(\pi \frac{a_{out}\omega_{out}}{\rho})} ,$$

where $\omega_{in} = \sqrt{k^2 + m^2}$, $\omega_{out} = \sqrt{(\frac{k}{a_{out}})^2 + m^2}$, and $\omega_- = \frac{1}{2}(a_{out}\omega_{out} - \omega_{in})$

- Remark 1. Universal Large k behavior: exponential decay (this happens for any smooth $a(t)$)

$$|\beta_k|^2 \sim_{k\to\infty} e^{-2\pi k/\rho}$$

- Remark 2. Adiabatic limit ($\rho \to 0$ behavior) [k fixed]. “The other side of the coin”

$$|\beta_k|^2 \sim_{\rho\to 0} e^{-2\pi k/\rho}$$

  The number density of created particles is an Adiabatic Invariant

- Remark 3. Massless limit: $m \to 0$.

$$|\beta_k|^2 \sim \frac{m^4}{k^2\rho^2} e^{-2\pi k/\rho} \to 0$$

  In agreement with conformal symmetry. No particle creation for $\xi = 1/6$, $m = 0$.

Vacuum persistence amplitude

- For a real scalar field [$|\alpha_k|^2 - |\beta_k|^2 = 1$]

$$\begin{aligned} |\langle 0_{out}|0_{in}\rangle|^2 &= \prod_{\vec{k}} |\alpha_k|^{-1} = \exp[-\frac{1}{2}\sum_{\vec{k}} \log(1+|\beta_k|^2)] \\ &= \exp[-\frac{V}{2(2\pi)^3}\int d^3k \log(1+|\beta_k|^2)] \\ &= \exp[-\frac{V}{2(2\pi)^3}\int d^3k \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n}|\beta_k|^{2n})] \end{aligned}$$

- For a charged (complex) scalar field

$$|\langle 0_{out}|0_{in}\rangle|^2 = \prod_{\vec{k}} |\alpha_k|^{-2} = \exp[-\frac{V}{(2\pi)^3}\int d^3k \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n}|\beta_k|^{2n})]$$

- For a Dirac field [spin $1/2$, $|\alpha_k|^2 + |\beta_k|^2 = 1$]

$$|\langle 0_{out}|0_{in}\rangle|^2 = \prod_{spin\ \vec{k}} |\alpha_k|^2 = \exp[-\frac{2V}{(2\pi)^3}\int d^3k \sum_{n=1}^{\infty} \frac{1}{n}|\beta_k|^{2n})]$$

Schwinger effect via frequency-mixing

- We want to see how the Schwinger effect is re-obtained via the frequency-mixing mechanism.
- Main idea: we can approach to an homogeneous and constant electric field

$$\vec{E} = (0, 0, -\mathcal{E}_0)$$

  by a limiting process
- Consider the so-called Sauter-pulse $(T > 0)$

$$\vec{E}(t) = (0, 0, -\mathcal{E}_0 \cosh^{-2}(t/T))$$

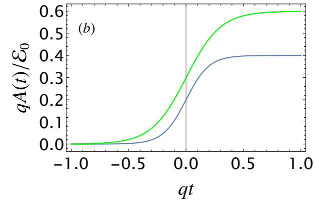
- In the limit $T \to \infty$ we approach the constant electric field

Sauter pulse

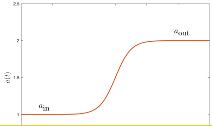
- The potential $A_\mu = (0,0,0,-A(t))$ is given by

$$A(t) = \mathcal{E}_0 \tau(\tanh(t/T) + 1)$$

which is bounded at early and late times, $A(-\infty) = A_{in} = 0$, $A(+\infty) = A_{out} = 2\mathcal{E}_0 T$.



- Similar to the case studied in the expanding universe: $a(t) \Leftrightarrow A(t)$



Particle creation via frequency-mixing

- Consider a charged scalar field (of charge e)

$$\phi(x) = \int d^3k \left[A_{\vec{k}} f_{\vec{k}}(\vec{x},t) + B^{\dagger}_{\vec{k}} f^{*}_{\vec{k}}(\vec{x},t)\right]$$

- $A_{\vec{k}}$ annihilate particles and $B^{\dagger}_{\vec{k}}$ create antiparticles
- Modes

$$f_{\vec{k}}(\vec{x},t) = \frac{1}{\sqrt{2(2\pi)^3}} e^{i\vec{k}\vec{x}} h_{\vec{k}}(t)$$

obeying

$$\ddot{h}_{\vec{k}}(t) + \left(m_{\perp}^2 + (k_3 - eA(t))^2\right) h_{\vec{k}}(t) = 0 \ ,$$

with

$$m_{\perp}^2 = m^2 + k_1^2 + k_2^2$$

Particle creation via frequency-mixing

- Asymptotic modes

$$f_{\vec{k}}(\vec{x},t) = \frac{1}{\sqrt{2(2\pi)^3}} h_{\vec{k}}^{in}(t) e^{i\vec{k}\vec{x}} \sim \frac{1}{\sqrt{2(2\pi)^3 \omega_{in}}} e^{i(\vec{k}\vec{x}-\omega_{in}t)} \qquad t \to -\infty \,,$$

$$g_{\vec{k}}(\vec{x},t) = \frac{1}{\sqrt{2(2\pi)^3}} h_{\vec{k}}^{out}(t) e^{i\vec{k}\vec{x}} \sim \frac{1}{\sqrt{2(2\pi)^3 \omega_{out}}} e^{i(\vec{k}\vec{x}-\omega_{out}t)} \qquad t \to +\infty \,,$$

where now

$$\omega_{in} = \sqrt{m_\perp^2 + (k_3 - eA_{in})^2}$$

$$\omega_{out} = \sqrt{m_\perp^2 + (k_3 - eA_{out})^2}$$

instead of $\omega_{in} = \sqrt{(k/a_{in})^2 + m^2}$ and $\omega_{out} = \sqrt{(k/a_{out})^2 + m^2}$.

Particle creation via frequency-mixing

- Beta coefficients for the charged scalar field

$$|\beta_{\vec{k}}|^2 = \frac{\cosh\left(\pi(\omega_{out} - \omega_{in})T\right) + \cosh\left(2\pi\kappa T\right)}{2\sinh\left(\pi\omega_{in}T\right)\sinh\left(\pi\omega_{out}T\right)} \ ,$$

- Beta coefficients for the charged Dirac field:

$$|\beta_{\vec{k}}|^2 = \frac{\cosh\left(2\pi q\mathcal{E}_0 T^2\right) - \cosh\left(\pi(\omega_{out} - \omega_{in})T\right)}{2\sinh\left(\pi\omega_{in}T\right)\sinh\left(\pi\omega_{out}T\right)} \leq 1$$

with

$$\omega_{in} = \sqrt{k_3^2 + m_\perp^2}$$

$$\omega_{out} = \sqrt{(k_3 - 2e\mathcal{E}_0 T)^2 + m_\perp^2}$$

and

$$\kappa = \frac{1}{2}\sqrt{(2e\mathcal{E}_0 T)^2 - (1/T)^2}$$

- Very important: as $T \to \infty$, the number density $|\beta_{\vec{k}}|^2$ behaves as (universal behaviour)

$$|\beta_{\vec{k}}|^2 \sim \Theta(k_3)\Theta(|eE_0|T - k_3)e^{-\pi \frac{k_1^2+k_2^2+m^2}{|eE_0|}}$$

Remark: After taking of the limit $T \to \infty$, $|\beta_{\vec{k}}|^2$ cannot be expanded as a power series in the coupling constant e

- For a charged scalar field the vacuum persistence amplitude can also be evaluated using the **frequency-mixing method**

$$|\langle 0_{out}|0_{in}\rangle|^2 \quad = \quad \prod_{\vec{k}} |\alpha_{\vec{k}}|^{-2} = \exp[-\frac{V}{(2\pi)^3}\int d^3k \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n}|\beta_{\vec{k}}|^{2n})]$$

- In the limit $T \to \infty$ we get

$$|\langle 0_{out}|0_{in}\rangle|^2 = \exp\left(-\frac{VT}{(2\pi)^3}(e\mathcal{E}_0)^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-n\pi m^2/(|e\mathcal{E}_0|)}\right) .$$

- This result is equivalent to the classical formula for the Schwinger effect (for charged bosons)

$$|\langle 0_{out}|0_{in}\rangle|^2 = e^{-2VT\Gamma_{Schwinger}} \quad \Gamma_{Schwinger} = \frac{e^2\mathcal{E}_0^2}{2(2\pi)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-n\pi m^2/(|e\mathcal{E}_0|)}$$

Vacuum persistence amplitude for charged Dirac fermions

- For a Dirac field the expression for the vacuum persistence amplitude is

$$|\langle 0_{out}|0_{in}\rangle|^2 \quad = \quad \prod_{spin\ \vec{k}} |\alpha_{\vec{k}}|^2 = \exp[-\frac{2V}{(2\pi)^3}\int d^3k \sum_{n=1}^{\infty} \frac{1}{n}|\beta_{\vec{k}}|^{2n})]$$

- Using also the asymptotic behavior $T \to +\infty$

$$|\beta_{\vec{k}}|^2 \sim \Theta(k_3)\Theta(|e\mathcal{E}_0|T - k_3)e^{-\pi\frac{k_1^2+k_2^2+m^2}{|e\mathcal{E}_0|}}$$

- We easily get, in the limit $\tau = T \to \infty$

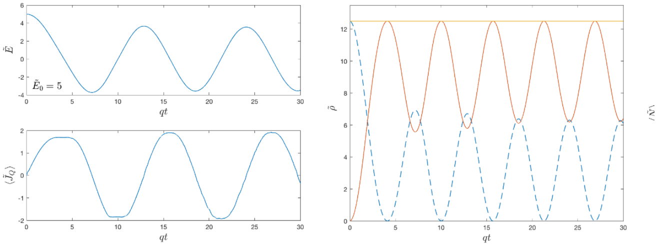
$$|\langle 0_{out}|0_{in}\rangle|^2 = \exp\left(-\frac{2VT}{(2\pi)^3}(e\mathcal{E}_0)^2 \sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi m^2/(|e\mathcal{E}_0|)}\right) .$$

- This reproduces the famous Schwinger's formula for charged fermions

$$|\langle 0_{out}|0_{in}\rangle|^2 = e^{-2VT\Gamma_{Schwinger}} \quad , \quad \Gamma_{Schwinger} = \frac{e^2\mathcal{E}_0^2}{(2\pi)^3 c\hbar}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi m^2c^2/(|e\hbar\mathcal{E}_0|)}$$

An aside: Schwinger pair creation with backreaction

- To solve the semiclassical Maxwell equations: $\partial_\mu F^{\mu\nu} = \langle J^\nu \rangle + J^\nu_{Classical}$ we need reliable control over $\langle J^\nu \rangle$. This expectation value $\langle J^\nu \rangle$ must be properly renormalized! [A. Ferreiro and J. Navarro-Salas, Phys. Rev. D **97**, no.12, 125012 (2018)]
- Various quantities are plotted for solutions to the Maxwell semiclassical backreaction equations (in $1+1$ dimensions) for a quantized spin- $\frac{1}{2}$ field with the classical current profile $J_C = -E_0\delta(t)$.



effect Backreaction.pdf

S. Pla, I. M. Newsome, R. S. Link, P. R. Anderson and J. Navarro-Salas, Phys. Rev. D **103**, no.10, 105003 (2021)

- The prediction that an expanding universe (or a time-dependent gravitational field) could spontaneously create particles was, and remains, a remarkable a surprising discovery.
- This phenomenon arises from the fact that the familiar creation and annihilation operators of quantum field theory evolve into superpositions of one another under the influence of the cosmic expansion.
- Remarkably, this effect is not an exotic anomaly but an inevitable consequence of two well-established frameworks: quantum field theory and general relativity.
- In words of Paul Davies (2007):

  It was a leap in the dark, and it turned out to be exactly what was needed. It took a lot of courage to embark on that. He [Parker] *could not have anticipated how important it would all turn out to be 10 or 15 years from then. He set the train in motion for a decade or more of similar work.*
- Parker's thesis is now accesible in the arXiv: 2507.05372 [gr-qc]